On the Expected Value of the Minimum Assignment

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Abstract

The minimum k-assignment of an $m \times n$ matrix X is the minimum sum of k entries of X, no two of which belong to the same row or column. If X is generated by choosing each entry independently from the exponential distribution with mean 1, then Coppersmith and Sorkin conjectured that the expected value of its minimum k-assignment is

$$\sum_{i,j \geq 0,\ i+j<k} \frac{1}{(m-i)(n-j)}$$

and they (with Alm) have proven this for $k \leq 4$ and in certain cases when $k = 5$ or $k = 6$. They were motivated by the special case of $k = m = n$, where the expected value was conjectured by Parisi to be $\sum_{i=1}^{k} \frac{1}{i^2}$. In this paper we describe our efforts to prove the Coppersmith–Sorkin conjecture. We give evidence for the following stronger conjecture, which generalizes theirs.

Conjecture Suppose that $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ are positive real numbers. Let X be a random $m \times n$ matrix in which entry x_{ij} is chosen independently from the exponential distribution with mean $\frac{1}{r_i c_j}$. Then the expected value of the minimum k-assignment of X is

$$\sum_{I,J} (-1)^{k-1-|I|-|J|} \cdot \binom{m+n-1-|I|-|J|}{k-1-|I|-|J|} \frac{1}{\left(\sum_{i \notin I} r_i\right) \cdot \left(\sum_{j \notin J} c_j\right)}.$$

Here the sum is over proper subsets I of $\{1, \ldots, m\}$ and J of $\{1, \ldots, n\}$ whose cardinalities $|I|$ and $|J|$ satisfy $|I| + |J| < k$.

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1 Problem Description and Background

Suppose that k, m and n are positive integers with $k \leq m \leq n$. A *minimum k-assignment* of an $m \times n$ matrix X is a set of k entries of X, no two of which belong to the same row or column, whose sum is as small as possible. We denote the value of this minimum sum by $\min_k(X)$.

We say that a random real number x is *exponentially distributed* with *rate a* if it is chosen according to the density ae^{-ax}, $x \geq 0$. The mean value of a rate a exponentially distributed quantity is $1/a$.

Suppose that we generate a random $m \times n$ matrix X by choosing each entry independently from the exponential distribution with rate 1. In [CS] Coppersmith and Sorkin conjectured that the expected value of its minimum k-assignment is

Conjecture 1

$$E(\min_k(X)) = \sum_{i,j \geq 0,\ i+j<k} \frac{1}{(m-i)(n-j)}. \tag{1}$$

In [AS] Alm and Sorkin show that this conjecture is correct when $k \leq 4$, when $k = m = 5$, and when $k = m = n = 6$.

The conjecture of Coppersmith and Sorkin generalized a conjecture of Parisi [P] who considered the case $k = m = n$. In this case, as shown in [CS], (1) reduces to

$$E(\min_k(X)) = \sum_{i=1}^{k} \frac{1}{i^2}. \tag{2}$$

In this paper we describe our efforts to prove these conjectures. Our main result is Conjecture 2, which generalizes the Coppersmith-Sorkin conjecture.

We will say that a matrix X is *random exponential* with *rate matrix $A = (a_{ij})$* if each entry x_{ij} is chosen independently according to the exponential distribution with rate a_{ij}. The expected value of the minimum k-assignment of such a matrix X is then a function of the rate matrix A. We denote this function by $E_k(A)$.

We will show that $E_k(A)$ is a rational function of the rates a_{ij} and give an explicit method for computing it, at least in principle. Then we will specialize to the case when the rate matrix has rank 1, for which we have the following explicit formula.

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Conjecture 2 *Suppose that $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ are positive real numbers and that $a_{ij} = r_i c_j$. Let X be a random $m \times n$ matrix in which entry x_{ij} is chosen independently from the exponential distribution with rate a_{ij}. Then the expected value of the minimum k-assignment of X is*

$$\sum_{I,J} (-1)^{k-1-|I|-|J|} \cdot \binom{m+n-1-|I|-|J|}{k-1-|I|-|J|} \frac{1}{\left(\sum_{i \notin I} r_i\right) \cdot \left(\sum_{j \notin J} c_j\right)}.$$

Here the sum is over proper subsets I of $\{1, \ldots, m\}$ and J of $\{1, \ldots, n\}$ whose cardinalities $|I|$ and $|J|$ satisfy $|I| + |J| < k$.

Example 1 *The expected value of the minimum 1-assignment of a random exponential matrix with rate matrix $a_{ij} = r_i c_j$ is*

$$\frac{1}{\left(\sum_i r_i\right) \cdot \left(\sum_j c_j\right)}.$$

Example 2 *The expected value of the minimum 2-assignment of a 3×3 random exponential matrix with rate matrix $a_{ij} = r_i c_j$ is*

$$\left(\frac{1}{r_2 + r_3} + \frac{1}{r_1 + r_3} + \frac{1}{r_1 + r_2}\right) \frac{1}{c_1 + c_2 + c_3}$$
$$+ \left(\frac{1}{c_2 + c_3} + \frac{1}{c_1 + c_3} + \frac{1}{c_1 + c_2}\right) \frac{1}{r_1 + r_2 + r_3}$$
$$- \frac{5}{(r_1 + r_2 + r_3)(c_1 + c_2 + c_3)}$$

We will provide evidence in support of Conjecture 2.

We also have a stronger conjecture for which we will provide evidence, although perhaps this evidence is not as strong as that for Conjecture 2. A matrix can have several minimum k-assignments for some value of k. However, with probability 1, a random matrix has a single minimum k-assignment for each k. Suppose that M is a $(k-1) \times (k-1)$ submatrix of X and that $\chi_M(X)$ is the function with value 1 when M contains a minimum $(k-1)$-assignment of X and 0 otherwise. Then we define the *expected contribution* of M to the minimum k-assignment of X as the expected value of the random variable $\chi_M(X) \min_k(X)$. It is clear that $E(\min_k(X))$ is the sum of the expected contributions of all the $(k-1) \times (k-1)$ submatrices. Our stronger conjecture gives a formula for the expected contribution of M when the rate matrix has rank 1.

Conjecture 3 *Suppose that $A = (r_i c_j)$ is a positive $m \times n$ matrix with rank 1 and that X is a random exponential matrix with rate matrix A. Let I be a set of $k-1$ elements of $\{1, \dots, m\}$, let J be a set of $k-1$ elements of $\{1, \dots, n\}$, and let M be the $(k-1) \times (k-1)$ submatrix of X with rows indexed by I and columns indexed by J. Then the expected value of $\chi_M(X) \cdot \min_k(X)$ is*

$$\sum_{i,j} \left(\prod_{t=1}^{k-1} \frac{r_{i_t} c_{j_t}}{\left(R - \sum_{s=1}^{t-1} r_{i_s}\right)\left(C - \sum_{s=1}^{t-1} c_{j_s}\right)} \right) \sum_{t,u \geq 0, t+u < k} \frac{1}{(R - \sum_{s=1}^{t} r_{i_s})(C - \sum_{s=1}^{u} c_{j_s})}$$

where the outer sum is over permutations $(i_1, \dots, i_{k-1})$ and $(j_1, \dots, j_{k-1})$ of I and J, respectively, and R and C denote the sums of all r_i's and all c_j's, respectively.

We shall see that Conjecture 3 implies Conjecture 2 and that Conjecture 2 in turn implies Conjecture 1.

Section 2 discusses what we know for the expected minimum assignment when the rate matrix is arbitrary.

In Section 3 we discuss the way we arrived at Conjecture 2 and give some equivalent formulations, one of which is directly implied by Conjecture 3.

We discuss the computational evidence for our conjectures in Section 4.

Section 5 gives additional evidence for Conjecture 2.

We would like to thank Jim Propp for bringing this problem to our attention.

2 Theory for a general rate matrix

2.1 Expected value for a general rate matrix

We begin by showing that the general formula for the expected value of the minimum assignment of a random exponential matrix is a rational function of the rates, with denominators factoring into linear terms of special form.

Recall that k, m, n are positive integers with $k \leq m \leq n$ and that $A = (a_{ij})$ is a positive $m \times n$ matrix. We form a random matrix X by choosing x_{ij} independently from the exponential distribution with rate a_{ij}. The expected value of the minimum k-assignment of X is then a function of A, which we will denote by $E_k(A)$.

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By definition of expected value, $E_k(A)$ is given by the integral expression

$$E_k(A) = \left(\prod_{i,j} a_{ij} \right) \int_{X \geq 0} \min_k(X) e^{-A \cdot X} dX$$

where the integral is taken over the space of all nonnegative matrices X. Here $A \cdot X$ denotes the dot product $\sum_{i,j} a_{ij} x_{ij}$ and dX denotes the product $\prod_{i,j} dx_{ij}$.

We denote by $\mathcal{S}_k$ the set of all $m \times n$ matrices σ such that all the entries of σ are 0's except for k entries which are 1's, no two in the same row or column. There are $k! \binom{m}{k} \binom{n}{k}$ such matrices in $\mathcal{S}_k$ and these we identify in the obvious way with the possible locations of the minimum k-assignment of X. In particular,

$$\min_k(X) = \min_{\sigma \in \mathcal{S}_k} (\sigma \cdot X) .$$

For each σ we denote by P_σ the set of nonnegative matrices X for which $\min_k(X) = \sigma \cdot X$; that is, P_σ is the set of nonnegative matrices X for which the minimum k-assignment is σ. Thus, we have

$$E_k(A) = \left(\prod_{i,j} a_{ij} \right) \sum_{\sigma \in \mathcal{S}_k} \left[\int_{X \in P_\sigma} (\sigma \cdot X) e^{-A \cdot X} dX \right] . \qquad (3)$$

Note that each of the sets P_σ is a polyhedral cone determined by a finite set of homogeneous linear inequalities $\sigma \cdot X \leq \tau \cdot X$ for all $\tau \in \mathcal{S}_k$. As a consequence, each P_σ can be decomposed into a finite collection $\mathcal{C}_\sigma$ of simplicial cones. It seems difficult to give an explicit description of $\mathcal{C}_\sigma$. Nevertheless, we can derive some useful properties of $E_k(A)$ from the fact that this decomposition exists. First we rewrite (3) as

$$E_k(A) = \left(\prod_{i,j} a_{ij} \right) \sum_{\sigma \in \mathcal{S}_k} \sum_{C \in \mathcal{C}_\sigma} \left[\int_C (\sigma \cdot X) e^{-A \cdot X} dX \right] .$$

Each cone C is the set of nonnegative linear combinations of a set of mn linearly independent vectors V_i, $i = 1, \ldots, mn$, where each V_i is a nonnegative $m \times n$ matrix. For the part of the integral over C, we make the substitution $X = \sum_i u_i V_i$, where $U = (u_1, \ldots, u_{mn})$ ranges over all nonnegative mn-

tuples. We can then explicitly compute the integral over C as

$$
\begin{aligned}
\int_C (\sigma \cdot X) e^{-A \cdot X} dX &= |\det V| \int_{U \geq 0} \sum_{i=1}^{mn} u_i (\sigma \cdot V_i) e^{-\sum_{j=1}^{mn} u_j A \cdot V_j} dU \\
&= |\det V| \sum_{i=1}^{mn} \left[(\sigma \cdot V_i) \int_{U \geq 0} u_i e^{-\sum_{j=1}^{mn} u_j A \cdot V_j} dU \right] \\
&= |\det V| \left(\sum_{i=1}^{mn} \frac{\sigma \cdot V_i}{A \cdot V_i} \right) \left(\prod_{i=1}^{mn} \frac{1}{A \cdot V_i} \right)
\end{aligned}
$$

where $|\det V|$ is the mn-volume of the parallelepiped determined by $V_1, \ldots, V_{mn}$. Thus, we obtain the expression

$$
E_k(A) = \left(\prod_{i,j} a_{ij} \right) \sum_{\sigma \in \mathcal{S}_k} \sum_{C \in \mathcal{C}_\sigma} |\det V| \left(\sum_{i=1}^{mn} \frac{\sigma \cdot V_i}{A \cdot V_i} \right) \left(\prod_{i=1}^{mn} \frac{1}{A \cdot V_i} \right). \quad (4)
$$

Note that although the vectors V_i depend on C and σ, they do not depend on A. Thus, $E_k(A)$ is a rational function of the a_{ij}'s, homogeneous of degree -1.

We can obtain more information about the rational function $E_k(A)$ by constructing, for each $\sigma \in \mathcal{S}_k$, a finite set of generators of P_σ in the sense that every element of P_σ is a nonnegative linear combination of the generators.

For this purpose we define two classes of matrices. First, for any $1 \leq i \leq m$ and $1 \leq j \leq n$ we define e_{ij} to be the matrix that is all zero except for a single 1 at position (i, j). Next, for any sets $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$, we define V_{IJ} to be the matrix obtained from the all 1's matrix by zeroing out all entries in the rows indexed by I and the columns indexed by J. It is easy to see that $\min_k(V_{IJ}) = \max(0, k - |I| - |J|)$. Thus V_{IJ} is in P_σ if and only if $\sigma \cdot V_{IJ} = \max(0, k - |I| - |J|)$.

Theorem 1 *Every element of P_σ is a nonnegative linear combination of e_{ij}'s with $e_{ij} \cdot \sigma = 0$ and V_{IJ}'s in P_σ with $|I| + |J| < k$.*

We prove Theorem 1 using a reduction procedure on the matrices of P_σ. Let X be a matrix in P_σ and suppose that $\min_k(X) = s$. We choose an arbitrary linear order for the e_{ij}'s and denote this ordered set by $e_1, e_2, \ldots, e_{mn}$. Then we choose a sequence of nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{mn}$ as

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follows. Once $\alpha_1, \ldots, \alpha_{i-1}$ are chosen we select α_i as large as possible so that $X - (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_i e_i)$ is nonnegative and has minimum k-assignment with value s.

Set
$$Y = X - (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{mn} e_{mn}).$$

Note that if $e_i \cdot \sigma \neq 0$, then we will have $\alpha_i = 0$, since otherwise $\sigma \cdot (X - (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_i e_i)) < s$. Thus X is Y plus a nonnegative linear combination of the e_{ij}'s given in Theorem 1.

We say that an entry y_{ij} of a matrix Y *participates* in a minimum k-assignment if there is a minimum k-assignment using the entry y_{ij}.

We say that a nonnegative matrix $Y = (y_{ij})$ is k-*reduced* if every nonzero entry of Y participates in a minimum k-assignment.

It is straightforward to see that the matrix Y resulting from our reduction process applied to $X \in P_\sigma$ is k-reduced and that $Y \in P_\sigma$. It remains to show that every k-reduced matrix Y with minimum k-assignment σ is a nonnegative linear combination of the appropriate V_{IJ}. This will require a series of preliminary results.

First we need a simple combinatorial lemma.

Lemma 1 *Suppose that T is a matrix all of whose entries are 0, 1, or 2 and whose row and column sums are at most 2, and that the sum of all the entries in T is $2k$. Then $T = \sigma + \tau$ for some k-assignments σ and τ.*

Proof: We may assume there are no 2's in T, since if there is a 2 we know that both σ and τ must have a 1 there, and only 0's everywhere else in its row and column. So we assume T is a 0-1 matrix whose row and column sums are at most 2, such that the sum of all entries is $2s$ for some $s \leq k$, and we want to find two s-assignments σ and τ such that $\sigma + \tau = T$. Identify T with a graph with vertices at each 1 of T, and edges between any two 1's belonging to the same row or column. Clearly every vertex of T has degree ≤ 2, so every component of T is a chain or a cycle. The vertices in each component can be alternately assigned to σ and τ. If there is an odd component (which must be a chain) there must be another odd component to balance it out (so one can have an extra σ vertex, and the other can have an extra τ vertex), since the sum of all entries in T is even. $\square$

Lemma 2 *Suppose that Y is a k-reduced matrix and*

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a submatrix of Y. If a and d each participate in a minimum k-assignment of Y, then $a + d \leq b + c$. If also $a + d = b + c$, then b and c also each participate in a minimum k-assignment. These statements also hold for a and d switched with b and c.

Proof: Suppose that $a + d > b + c$. Let σ_1 and τ_1 be minimum assignments passing through a and d respectively. Form the matrix $T_1 = \sigma_1 + \tau_1$. Then form T from T_1 by subtracting 1 at the positions of a and d and adding 1 at the positions of b and c. The hypotheses of Lemma 1 apply to T so that $T = \sigma + \tau$ for some k-assignments σ and τ. But since $a + d > b + c$ we must have

$$\sigma_1 \cdot Y + \tau_1 \cdot Y = T_1 \cdot Y > T \cdot Y = \sigma \cdot Y + \tau \cdot Y,$$

contradicting the minimality of the assignments σ_1 and τ_1. Thus $a + d \leq b + c$.

Now suppose that $b + c = a + d$. Then the same construction yields

$$\sigma_1 \cdot Y + \tau_1 \cdot Y = T_1 \cdot Y = T \cdot Y = \sigma \cdot Y + \tau \cdot Y,$$

so that both σ and τ are minimum k-assignments, and at least one includes b and at least one includes c.

This proof obviously also holds with a and d switched with b and c. □

Proposition 1 *Suppose that $Y = (y_{ij})$ is a k-reduced $m \times n$ matrix. Then there exist $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ such that*

$$y_{ij} = \max(0, \lambda_i + \mu_j) \tag{5}$$

and such that y_{ij} participates in a minimum k-assignment precisely when $y_{ij} = \mu_i + \lambda_j$.

Proof: Let $d = y_{tu}$ denote the largest entry in Y. Take λ_i to be the ith entry in the column of d, so that $\lambda_i = y_{iu}$. Let μ_j to be the jth entry in the row of d, decreased by d, so $\mu_j = y_{tj} - d$.

First we prove (5).

When y_{ij} is in the row or column of d, then we have $y_{ij} = \lambda_i + \mu_j$, so (5) is immediate.

Suppose that a is in neither the row of d nor the column of d. Let

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the submatrix of Y containing the rows and columns of a and d (where the order of the rows or columns in S may be opposite to the order they occur in Y). Then we just need to show that $a = \max(0, b + c - d)$.

First suppose that $b + c - d > 0$. Then b, c, d must all be positive, because d is maximum for the whole matrix. Since b, c are both positive, they must both participate in a minimum k-assignment, so $b + c \leq a + d$, by Lemma 2. But then, $a \geq b + c - d$, so a is positive. Then a and d both participate in a minimum k-assignment, which implies $a + d \leq b + c$, so $a = b + c - d$.

Next, suppose that $b + c - d \leq 0$. If $a > 0$, then a and d both participate in a minimum k-assignment, so $d < a + d \leq b + c$, a contradiction. Thus $a = 0$.

Now we show that $y_{ij} = \lambda_i + \mu_j$ exactly when y_{ij} participates in a minimum k-assignment.

Recall that for any y_{ij} in either the row or column of d, we have $y_{ij} = \lambda_i + \mu_j$. So we need to show that all such entries participate in a minimum k-assignment. Let b be any entry in the column of d. We already know that positive entries must participate in a minimum k-assignment, so we assume that $b = 0$. If $d = 0$, then our whole matrix is zero and our result is trivial, so we may assume that $d > 0$ and therefore participates in a minimum k-assignment. If the minimum assignment using d does not use the row of b, we can replace d by b and obtain a smaller assignment, a contradiction. So we can conclude that the minimum assignment using d also uses an element a from the row of b. Form the 2×2 submatrix S containing a and d as above. Since d is the largest entry, $c \leq d$, and we also have $b = 0 \leq a$. Thus we can exchange a and d for b and c, to obtain a minimum k-assignment in which b participates. In the same way, we see that any entry in the row of d must participate in a minimum k-assignment.

Finally, consider an element a that is neither in the row nor the column of d and form the 2×2 submatrix S containing a and d as above. We must show that a participates in a minimum assignment exactly when $a = b + c - d$. This is certainly true if $a > 0$. So, let us assume that $a = 0$. Also, since

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both b and c are in a row or a column of d, both participate in a minimum assignment, so that $b + c \leq a + d$.

Now suppose that a participates in a minimum k-assignment. Then $a + d \leq b + c$, so $a = b + c - d$, as required.

Conversely, suppose that $a = b + c - d$. Then, since b and c participate in minimum k-assignments, Lemma 2 shows that a also participates. $\square$

Proof of Theorem 1: Now we are ready to prove Theorem 1 by showing that any k-reduced matrix Y in P_σ is a nonnegative linear combination of a suitable collection of matrices V_{IJ} from P_σ. Without loss of generality we can assume that the λ's and μ's are weakly increasing. In this case the rows and columns of Y are also weakly increasing.

If the matrix Y is zero, there is nothing to prove. Otherwise, since every nonzero entry in Y participates in a minimum k-assignment, we know that the minimum k-assignment is nonzero. Hence there is at least one nonzero entry among $y_{1,k}, y_{2,k-1}, \ldots, y_{k,1}$. In particular, there is a pair (i, j) such that $y_{ij} > 0$ and $i + j \leq k + 1$. Now select such a pair (i, j) to be minimal in the sense that if $i \neq 1$ then $y_{i-1,j} = 0$ and if $j \neq 1$ then $y_{i,j-1} = 0$. Let $I = \{1, \ldots, i-1\}$ and let $J = \{1, \ldots, j-1\}$. We will show that V_{IJ} is in P_σ and that $Y - y_{ij}V_{IJ}$ is again in P_σ and still k-reduced.

Suppose that $1 \leq i' < i$ and $1 \leq j' < j$. Then, since

$$\max(0, \lambda_{i'} + \mu_j) = y_{i'j} = 0 < y_{ij} = \max(0, \lambda_i + \mu_j),$$

we know that $\lambda_{i'} < \lambda_i$. Similarly $\mu_{j'} < \mu_j$. Since $y_{i'j} = \max(0, \lambda_{i'} + \mu_j) = 0$, we have $\lambda_{i'} + \mu_j \leq 0$. But then

$$\lambda_{i'} + \mu_{j'} < \lambda_{i'} + \mu_j \leq 0.$$

If follows from Proposition 1 that none of the matrix entries $y_{i'j'}$ with $i' < i$ and $j' < j$ can participate in a minimum k-assignment.

To see that $V_{IJ} \in P_\sigma$, first note that any minimum k-assignment of Y must use all of the first $i - 1$ rows. If not, since $i \leq k$, there is some $i_1 > i$ such that row i_1 participates. Then, since not all of the first $i - 1$ rows are used, we can replace the entry of the assignment in row i_1 with the entry in the same column of row i_0, for some $i_0 < i$, to get an assignment with a value no larger. The entry being replaced could not come from a column preceding j, by the discussion above, so it must be positive. But also by the discussion above, $\lambda_{i_0} < \lambda_{i_1}$. Then the new assignment would be strictly

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smaller, a contradiction. Thus, any minimum assignment uses all of the first $i-1$ rows and all of the first $j-1$ columns, and does not use any entry which is in both the first $i-1$ rows and the first $j-1$ columns. So, if τ is any matrix representing a minimum k-assignment of Y, we must have $V_{IJ} \cdot \tau = k - |I| - |J|$. In particular, $V_{IJ} \cdot \sigma = k - |I| - |J|$, so V_{IJ} is in P_σ as claimed.

The preceding argument shows that if we replace Y by $Y' = Y - tV_{IJ}$, for any t satisfying $0 \le t \le y_{ij}$, the effect on any minimum k-assignment is to subtract $(k - |I| - |J|)t$ from its value. Thus, all assignments τ that are minimum for Y will agree on Y'. We now show that each of these assignments τ is minimum for Y' as well. Assume not. Then we could find t_1 and t_2 such that $0 \le t_1 < t_2 \le y_{ij}$ and k-assignments τ and ϕ such that τ is minimum for $(Y - tV_{IJ})$ when $t \le t_1$ but not when $t_1 < t \le t_2$, and ϕ is minimum for $Y - tV_{IJ}$ when $t_1 \le t \le t_2$. Thus, we would have both ϕ and τ minimum assignments for $Y - t_1 V_{IJ}$. But then our preceding argument applied to $Y - t_1 V_{IJ}$ tells us that ϕ and τ must agree on $Y - tV_{IJ}$ for $t_1 \le t \le t_2$, a contradiction.

Now we let $Y' = Y - y_{ij}V_{IJ}$ and observe that Y' is k-reduced. Indeed, all the assignments τ that were minimum for Y are also minimum for Y'. Thus, any element that participated in a minimum k-assignment for Y will also participate in a minimum k-assignment for Y'. Also, the replacement of Y by Y' creates no new nonzero elements, so the matrix Y' will be k-reduced.

Since σ in particular is a minimum assignment for Y, σ will also be minimum for Y', so that Y' is also in P_σ.

Note that Y' has at least one more zero entry than Y, namely the entry at (i, j).

We can continue removing multiples of submatrices V_{IJ}, each time producing a matrix with at least one more zero entry. Thus, we eventually reach a matrix that is all zero. In effect, we have expressed Y as a nonnegative linear combination of the generators as required. $\square$

We have shown that every element of P_σ is a nonnegative linear combination of certain e_{ij}'s and V_{IJ}'s in P_σ. We remark that the e_{ij}'s generate extreme rays of P_σ but the V_{IJ}'s in general do not. It is not hard to show that the V_{IJ}'s which do generate extreme rays of P_σ are those with $|I|+|J| = k-1$.

We now observe that Theorem 1 allows us to make some conclusions about the rational function $E_k(A)$. A simplicial cone in a decomposition of P_σ has some generators of the form e_{ij} and some of the form V_{IJ}. For the

generators of the form e_{ij}, we know that $\sigma \cdot e_{ij} = 0$. Also the dot products $e_{ij} \cdot V$ in the denominator cancel with factors in the initial product of a_{ij}'s in (4). Thus, the denominator of the integral over a simplicial cone is a product of terms of the form $A \cdot V_{IJ}$. Finally, we can conclude:

Theorem 2 *The expected minimum k-assignment of a random exponential matrix with rate matrix A is a rational function of the entries of A. The denominator of the rational function is a product of sums, each being the sum of all entries in a submatrix of A omitting i rows and j columns, where $i + j < k$.*

Example 3 *The minimum 2-assignment of a random 2×2 exponential matrix with rate matrix $A = (a_{ij})$ is*

$$\frac{1}{a_{11} + a_{12}} + \frac{1}{a_{21} + a_{22}} + \frac{a_{11}a_{21}}{(a_{11} + a_{12})(a_{21} + a_{22})(a_{12} + a_{22})}$$
$$+ \frac{a_{12}a_{22}}{(a_{11} + a_{12})(a_{21} + a_{22})(a_{11} + a_{21})}$$

The above formula is easily computed by the method we will sketch in Section 4.

In the case of a rank 1 rate matrix $A = (r_i c_j)$, Theorem 2 says that $E_k(A)$ is a rational function of the r_i's and c_j's and its denominator is a product of sums of subsets of the r_i's omitting fewer than k of the r_i's and sums of subsets of the c_j's omitting fewer than k of the c_j's. Since this is a consequence of Conjecture 2, it lends some support to the conjecture.

2.2 The nesting lemma

A real nonnegative $m \times n$ matrix X has minimum k-assignments for each $k \leq m$. Generically there is only one of each but in some cases there are many minimum assignments of various sizes. It helps to know how these are related.

The following lemma is fundamental. Other proofs probably exist but we include ours here for completeness.

Lemma 3 *Let k_1 and k_2 be two integers, with $k_1 \leq k_2 \leq m$.*

Suppose that M_1 is a $k_1 \times k_1$ submatrix of X that contains a minimum k_1-assignment of X. Then there exists a $k_2 \times k_2$ submatrix M_2 containing M_1 such that M_2 contains a minimum k_2-assignment of X.

Suppose that M_2 is a $k_2 \times k_2$ submatrix of X that contains a minimum k_2-assignment of X. Then there exists a $k_1 \times k_1$ submatrix M_1 contained in M_2 such that M_1 contains a minimum k_1-assignment of X.

Proof: If $k_1 = k_2$, there is nothing to prove. So assume $k_1 < k_2$ and fix a minimum k_1-assignment and a minimum k_2-assignment.

Let G be the graph on k_1 red vertices (representing the entries of the k_1-assignment) and k_2 blue vertices (representing the entries of the k_2-assignment), with edges between two vertices if the corresponding entries belong to the same row or column. (If the assignments share an entry then we have a red vertex and a blue vertex with two edges between them comprising a component which is a cycle of length 2.)

Then G is bipartite and no vertex of G has degree more than 2. Thus, every component of G is a cycle or a chain in which the red and blue vertices alternate.

Suppose some component of G has m_1 red vertices and m_2 blue vertices, and all of its red vertices have degree 2. Such a component is either a cycle, so that $m_1 = m_2$, or a chain with blue vertices at each end, so that $m_1 + 1 = m_2$. In either case we have $m_1 + 1 \geq m_2$.

Consider the submatrix M_1 spanned by the associated m_1 entries of the k_1-assignment and the submatrix M_2 spanned by the associated m_2 entries of the k_2-assignment. The component condition translates into the condition that the M_1 is contained in M_2. Also the remaining entries of the two assignments comprise a minimum $(k_1 - m_1)$-assignment and a minimum $(k_2 - m_2)$-assignment of the submatrix of X complementary to M_2. Since $k_1 < k_2$ and $m_1 + 1 \geq m_2$, we have $k_1 - m_1 \leq k_2 - m_2$, so the lemma follows by induction applied to the complementary submatrix.

Now assume that every component of G has a red vertex of degree 1 or less. Such components are chains with one endpoint red. When the other endpoint is red, there are more red than blue vertices in the component. When the other endpoint is blue, the number of vertices of both colors is equal. In particular the number of red vertices is always at least as great as the number of blue vertices. It follows that, all together, there are at most k_1 blue vertices that are connected to some red vertex. Thus we can select $k_2 - k_1$ entries of the k_2-assignment that do not share any row or

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column with the k_1-assignment. We can now consider three sets of matrix entries, the set S_1 of k_1 entries of the k_1-assignment, the set S_2, just selected, of $k_2 - k_1$ of entries of the k_2-assignment and the set S_3 of the remaining k_1 entries of the k_2-assignment. Then the sets S_3 and S_1 are both in the submatrix complementary to that determined by S_2. Moreover the sums of the entries in S_1 and S_3 must be equal. For, if the sum of the S_1 entries were greater than that of S_3, then S_3 would be a smaller k_1-assignment than S_1, a contradiction. But if the sum of the S_1 entries were smaller than the sum of S_3 entries, then $S_1 \cup S_2$ would be give a smaller k_2-assignment than $S_2 \cup S_3$, which was a minimum k_2-assignment.

Thus S_3 is a minimum k_1-assignment contained in our minimum k_2-assignment and $S_1 \cup S_2$ is a minimum k_2-assignment containing our minimum k_1-assignment, which proves our lemma. □

2.3   A computational consequence of the nesting lemma

Lemma 3 suggests a way to compute the minimum k-assignment of a matrix X when $k < n$. We proceed by finding the minimum k-assignments for X for $k = 1, 2, \ldots$, one at a time. Suppose we have found a minimum $(k-1)$-assignment whose rows and columns determine a $(k-1) \times (k-1)$ submatrix M of X. Then, when we search for a minimum k-assignment, we know that we can restrict our search to minimum k-assignments in one of the submatrices of X obtained by appending a single new row and new column to M.

There are some simple properties that such an extension must have. Suppose the submatrix of a minimum k-assignment uses M together with a new row i and a new column j. Also, suppose that the minimum k-assignment uses an entry from column j that is in row i' of M. Then this entry in column j must be the smallest entry in the part of row i' outside of M. Similarly, if the minimum k-assignment uses an entry from row i that is in column j' of M, then this entry in row i must be the smallest entry in the part of column j' outside M. Finally, if the minimum k-assignment uses the entry x_{ij}, then this entry must be minimum in the submatrix of X complementary to M.

The preceding discussion shows that the following strategy will work to construct the minimum k-assignment once we have found the minimum $(k-1)$-assignment. We define the $k \times k$ *auxiliary matrix* $\text{Aux}_M(X)$ by appending to M a new row and column as follows. To each row i of M we append a new entry which is the minimum of the entries in row i of X that are outside of M. To each column j of M we append a new entry which is the minimum

of the entries in column j of X that are outside of M. At the intersection of the new row and new column we place the minimum entry of the submatrix of X complementary to M.

We now find the minimum k-assignment of $\mathrm{Aux}_M(X)$. This assignment will use an entry in the last row and an entry in the last column of $\mathrm{Aux}_M(X)$, which can be the same entry if the assignment uses the entry in the last row and column. Each of these entries is a copy of some entry of X, which then tells us which row and column of X need to be appended to M to obtain the submatrix of the minimum k-assignment of X.

2.4   A consequence for the expected contribution

The discussion of the auxiliary matrix in the preceding section can be formalized to prove an interesting property of the expected contribution.

We will use the following simple facts about independent exponential random variables.

Proposition 2 *Suppose that $a_1, \ldots, a_m$ are positive real numbers, and that $x_1, \ldots, x_m$ are independent random variables with x_i chosen from the exponential distribution with rate a_i.*

Let x denote the random variable $\min_i x_i$. Then x is distributed as an exponential random variable of rate $a_1 + \cdots + a_m$.

Let W be the discrete random variable whose value is the least i for which $x = x_i$. Then W and x are independent random variables, and the probability that $W = i$ is $a_i/(a_1 + \cdots + a_m)$.

Proof: Let c be a positive real number. Then the probability that $x \geq c$ is

$$a_1 \cdots a_m \int_{x_1=c}^{\infty} \cdots \int_{x_m=c}^{\infty} e^{-\sum_{j=1}^{m} a_j x_j} dx_1 \cdots dx_m.$$

$$= a_1 \cdots a_m e^{-c\sum_{j=1}^{m} a_j} \int_{u_1=0}^{\infty} \cdots \int_{u_m=0}^{\infty} e^{-\sum_{j=1}^{m} a_j u_j} du_1 \cdots du_m.$$

$$= e^{-c\sum_{j=1}^{m} a_j},$$

where we have made the substitution $x_i = u_i + c$ on the second line. Taking the derivative with respect to c we see that x is distributed as an exponential random variable of rate $a_1 + \cdots + a_m$.

A similar substitution yields the part of the integral corresponding the event that $x = x_1$, as follows:

$$a_1 \cdots a_m \int_{x_1=c}^{\infty} \int_{x_2=x_1}^{\infty} \cdots \int_{x_m=x_1}^{\infty} e^{-\sum_{j=1}^{m} a_j x_j} dx_1 \cdots dx_m.$$

$$= a_1 \cdots a_m \int_{u_1=0}^{\infty} \cdots \int_{u_m=0}^{\infty} e^{-(a_1(c+u_1)+a_2(c+u_1+u_2)+\cdots a_m(c+u_1+u_m))} du_1 \cdots du_m$$

$$= \frac{a_1}{\sum_{j=1}^{m} a_j} e^{-c \sum_{j=1}^{m} a_j}.$$

Thus, the conditional probability that $x = x_i$, given $x \geq c$, is $a_i/(a_1 + \cdots + a_m)$. Since this is true for all c, the event $x = x_i$ is independent of the random variable x. $\square$

Now let X be any nonnegative $m \times n$ matrix and associate to X the $k \times k$ matrix $Y = \mathrm{Aux}_M(X)$ where M is the upper left $(k-1) \times (k-1)$ submatrix of X. For any $t < k$, let M_t denote the upper left $t \times t$ submatrix of X. By abuse of notation we will also let M_t denote the upper left $t \times t$ submatrix of Y, since they are identical.

Lemma 4 *For any $t < k$, let σ be a t-assignment of M_t. Then σ is a minimum t-assignment for X if and only if it is a minimum t-assignment for Y.*

Proof: Let σ be a t-assignment of M_t.

If $t = 1$ it is clear that the lemma holds. We proceed by induction on t.

Suppose that σ is a minimum t-assignment of X. By induction on t, there is a minimum $(t-1)$-assignment of Y which lies in M_t, so by Lemma 3, there is some minimum t-assignment τ of Y which uses at most one column and one row outside of M_t. By definition, each entry of Y outside of M_t equals some entry of X outside of M_t. If we replace each of the entries of τ outside of M_t by their equivalent entries in X, then we get a t-assignment τ' of X. Furthermore, $\tau' \cdot X \geq \sigma \cdot X$, since σ is minimum. But $\tau' \cdot X = \tau \cdot Y$ and $\sigma \cdot X = \sigma \cdot Y$, so $\tau \cdot Y \geq \sigma \cdot Y$. Therefore, σ is a minimum t-assignment of Y.

Now suppose that σ is a minimum t-assignment of Y. By induction on t, there is a minimum $(t-1)$-assignment of X which lies in M_t, so by Lemma 3, there is some minimum t-assignment τ of X which uses at most one column and one row outside of M_t. We can replace each entry x_{ij} of τ by the entry $y_{\min(i,k),\min(j,k)}$ to get a t-assignment τ' of Y. Furthermore, $\tau' \cdot Y \geq \sigma \cdot Y$,

since σ is minimum. But $\tau \cdot X \geq \tau' \cdot Y$ and $\sigma \cdot X = \sigma \cdot Y$, so $\tau \cdot X \geq \sigma \cdot X$. Therefore, σ is a minimum t-assignment of X. $\square$

Lemma 5 *The minimum $(k-1)$-assignment of X uses its first $k-1$ rows and columns exactly when the minimum $(k-1)$-assignment of Y uses its first $k-1$ rows and columns. In this case, the value of the minimum k-assignment of X is the same as the value of the minimum k-assignment of Y.*

Moreover in this case there is a minimum k-assignment τ of X and a minimum k-assignment τ' of Y which correspond entry by entry; i.e., the entries of τ in M are located in the same positions as the entries of τ' in M, and for any entry x_{ij} of τ outside of M there is a corresponding entry $y_{\min(i,k),\min(j,k)}$ of τ' outside of M.

Proof: The first paragraph is the case $t = k - 1$ in Lemma 4. The second paragraph follows from our discussion of auxiliary matrix in the preceding section. $\square$

We will use the more detailed statement in the second paragraph to prove Lemma 6 in Section 3.2.

Now let X be a random exponential $m \times n$ matrix with rate matrix A, and associate to A the $k \times k$ matrix $B = (b_{ij})$ with

$$b_{ij} = a_{ij} \quad \text{when } 1 \leq i, j \leq k - 1$$

and

$$b_{ik} = \sum_{j' \geq k} a_{ij'}, \quad i = 1, \ldots, k - 1$$

and

$$b_{kj} = \sum_{i' \geq k} a_{i'j}, \quad j = 1, \ldots, k - 1$$

and

$$b_{kk} = \sum_{i', j' \geq k} a_{i'j'}.$$

Note that $Y = \mathrm{Aux}_M(X)$ is a random exponential matrix with rate matrix B, by Proposition 2.

The statement in the first paragraph of the preceding Lemma shows that the expected contribution of the submatrix of X consisting of its first $k - 1$ rows and first $k - 1$ columns to the minimum k-assignment of X is the same

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as the expected contribution of the submatrix of Y consisting of its first $k-1$ rows and first $k-1$ columns to the minimum k-assignment of Y.

In the rank 1 rate matrix case, if $C(k, r_1, \ldots, r_m, c_1, \ldots, c_n)$ denotes the expected contribution of the submatrix of the first $k-1$ rows and columns when the rate matrix is $A = (r_i c_j)$, then

$$C(k, r_1, \ldots, r_{k-1}, r_k + \cdots + r_m, c_1, \ldots, c_{k-1}, c_k + \cdots + c_n)$$

is the expected contribution when the rate matrix is B. These two functions must agree, so the entries of A outside of the first $k-1$ rows and columns enter into the contribution function only via the sums defining the entries in the kth row and column of B.

Now note that Conjecture 3 has the feature that it predicts the same contribution in the two cases above. So this is a bit of evidence in favor of Conjecture 3.

Thus, we can summarize our discussion in the rank 1 rate matrix case as follows:

Theorem 3 *Let k, m and n be integers $k \leq m \leq n$. Then Conjecture 3 holds for k-assignments in an $m \times n$ matrix if and only if it holds for k-assignments in a $k \times k$ matrix.*

Thus, if one could prove only the cases $k = m = n$ of Conjecture 3, that would prove the general case of that conjecture as well as Conjecture 2 and Conjecture 1.

3 Rank 1 rate matrices

In this section we restrict our attention to random exponential matrices for which the rate matrix has rank 1. We describe how we arrived at Conjecture 2 and then give equivalent formulations, which provide different kinds of confirmation. Finally, we prove a probability result about the locations of the minimum ℓ-assignments for $1 \leq \ell \leq k$.

We discovered Conjecture 2 while experimenting with the computations described in Section 4. Finding that Mathematica had trouble carrying out the computation when the rate matrix consisted of mn indeterminates, we decided to try a simpler case, with rate matrix of the form $a_{ij} = a_i$ for all i and j. We noticed in this case that the answer has a surprisingly simple form—in

particular, it can be written as a linear combination of the reciprocals of sums of the a_i's. Next we found that, when the rate matrix has rank 1, so that $a_{ij} = r_i c_j$, the expected value seems to be a linear combination of terms of the form

$$\frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)} \tag{6}$$

with I a proper subset of $\{1, \ldots, m\}$, and J a proper subset of $\{1, \ldots, n\}$. It is easily shown that the rational functions $\frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}$ are linearly independent over the real numbers. Thus the coefficients in such a linear combination are uniquely determined.

Making the assumption that the expected value is indeed a linear combination of terms of the form (6), we arrived at Conjecture 2 by considering certain limiting conditions on the expected value. We will describe these limiting conditions in Section 5.

3.1 Equivalent formulations of Conjecture 2

From now on we use the shorthand notation $[m]$ for the set $\{1, \ldots, m\}$.

Let us introduce the notation

$$F(k, r, c) = \sum_{I,J} (-1)^{k-1-|I|-|J|} \cdot \binom{m+n-1-|I|-|J|}{k-1-|I|-|J|} \frac{1}{(\sum_{i \notin I} r_i) \cdot (\sum_{j \notin J} c_j)} \tag{7}$$

for the formula given in Conjecture 2. Here recall that $r = (r_1, \ldots, r_m)$ is an m-tuple of positive real numbers, $c = (c_1, \ldots, c_n)$ is an n-tuple of positive real numbers, and the sum is over proper subsets $I \subsetneq [m]$ and $J \subsetneq [n]$. The binomial coefficient enforces the condition $|I| + |J| < k$. In what follows we will often not mention such constraints explicitly.

In this section we derive alternative ways to write (7) and conclude that $F(k, r, c)$ is positive, Conjecture 2 implies Conjecture 1, and Conjecture 3 implies Conjecture 2.

Note that $F(k, r, c)$ can be written more succinctly as

$$F(k, r, c) = \sum_{I,J} \binom{k-1-m-n}{k-1-|I|-|J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}, \tag{8}$$

using binomial coefficients with negative numerator.

Proposition 3

$$F(k, r, c) = \sum_{|I'|+|J'|<k, I\subseteq I', J\subseteq J'} (-1)^{|I'|-|I|+|J'|-|J|} \frac{1}{(\sum_{i\notin I} r_i) \cdot (\sum_{j\notin J} c_j)}. \qquad (9)$$

Here I' and I are proper subsets of $[m]$ and J' and J are proper subsets of $[n]$.

Proof: Comparison with (8) shows that, for a fixed I and J with $|I|+|J| < k$, we need to evaluate the sum

$$\sum_{I\subseteq I', J\subseteq J', |I'|+|J'|<k} (-1)^{|I'|-|I|+|J'|-|J|}.$$

If we denote by i and j the cardinalities of I and J and by t and u the cardinalities of I' and J', we can rewrite this sum as

$$\sum_{t\geq i, u\geq j, t+u<k} (-1)^{t-i+u-j} \binom{m-i}{t-i} \binom{n-j}{u-j}$$

$$= \sum_{t\geq 0, u\geq 0, t+u<k-i-j} (-1)^{t+u} \binom{m-i}{t} \binom{n-j}{u}$$

$$= \sum_{l=0}^{k-1-i-j} (-1)^l \sum_{t\geq 0, u\geq 0, t+u=l} \binom{m-i}{t} \binom{n-j}{u}$$

$$= \sum_{l=0}^{k-1-i-j} (-1)^l \binom{m+n-i-j}{l}$$

$$= \sum_{l=0}^{k-1-i-j} \binom{i+j-m-n+l-1}{l}$$

$$= \sum_{l=0}^{k-1-i-j} \left(\binom{i+j-m-n+l}{l} - \binom{i+j-m-n+l-1}{l-1} \right)$$

$$= \binom{k-1-m-n}{k-1-i-j}$$

which agrees with (8). $\square$

We can rewrite (9) as a double sum with the inner sum over I and J and the outer sum over I' and J'. Then, for fixed I' and J', the inner sum factors as

$$\left(\sum_{I \subseteq I'} (-1)^{|I'|-|I|} \frac{1}{\sum_{i \notin I} r_i}\right) \left(\sum_{J \subseteq J'} (-1)^{|J'|-|J|} \frac{1}{\sum_{j \notin J} c_j}\right) \tag{10}$$

Now we show that each factor has an interesting probabilistic interpretation.

Suppose that an urn contains m balls labeled $1, 2, \ldots, m$ and for each i, ball i has weight r_i. We select balls one at a time without replacement, at each time selecting a ball with probability proportional to the weights of those balls still in the urn. Let $\Pr(r, I')$ denote the probability that the set of balls in I' are the first t balls to be chosen, where t is the cardinality of I'. Then

$$\Pr(r, I') = \sum_{\pi} \prod_{i=1}^{t} \frac{r_{\pi_i}}{R - \sum_{j=1}^{i-1} r_{\pi_i}} \tag{11}$$

where $R = \sum_{i=1}^{m} r_i$ and the outer sum is over all $t!$ orderings $(\pi_1, \ldots, \pi_t)$ of I'.

We can calculate $\Pr(r, I')$ in a different way as follows. Suppose we draw all m balls from the urn. If we fix any subset U of balls, then the probability that a particular ball u from U is chosen before any other ball from U is the weight of u divided by the sum of the weights of the balls in U.

Now, for $i \in I'$, let E_i denote the event that the first time a ball is drawn from the set consisting of i together with the complement of I', the ball chosen is from the complement of I'. Then E_i has probability

$$\frac{\sum_{j \notin I'} r_j}{r_i + \sum_{j \notin I'} r_j}.$$

In order for our set I' to be the set of the first t balls chosen, it is necessary and sufficient that none of the events E_i occur. For any subset I of I' the probability that all of the events E_i, $i \in I$, occur is

$$\frac{\sum_{j \notin I'} r_j}{\sum_{i \notin (I'-I)} r_i}$$

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So, by the Inclusion-Exclusion Principle,

$$\Pr(r, I') = \sum_{I \subseteq I'} (-1)^{|I'|-|I|} \frac{\sum_{i \notin I'} r_i}{\sum_{i \notin I} r_i} \tag{12}$$

which is $(\sum_{i \notin I'} r_i)$ times the first factor in (10). The analogous result holds for the second factor in (10). We conclude that

Proposition 4

$$F(k, r, c) = \sum_{I, J, |I|+|J|<k} \frac{\Pr(r, I) \Pr(c, J)}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}, \tag{13}$$

where the sum is over proper subsets I of $[m]$ and J of $[n]$. Hence $F(k, r, c)$ is always positive.

□

Now we rewrite $F(k, r, c)$ to show that Conjecture 2 implies Conjecture 1 and Conjecture 3 implies Conjecture 2.

Let $\Pr(r, (i_1, \ldots, i_t))$ denote the probability that the first t selections from our urn are $i_1, \ldots, i_t$ in that order. Then, from (13) and (11), we get

$$F(k, r, c) = \sum_{t, u \geq 0, t+u < k} \sum_{i, j} \frac{\Pr(r, (i_1, \ldots, i_t)) \Pr(c, (j_1, \ldots, j_u))}{(R - \sum_{s=1}^{t} r_{i_s})(C - \sum_{s=1}^{u} c_{j_s})}, \tag{14}$$

where the sum is over all sequences $i = (i_1, \ldots, i_t)$ of distinct integers in $[m]$ and $j = (j_1, \ldots, j_u)$ of distinct integers in $[n]$. But, certainly

$$\Pr(r, (i_1, \ldots, i_t)) = \sum_{i} \Pr(r, (i_1, \ldots, i_t, \ldots, i_{k-1}))$$

where the sum is over all extensions of $(i_1, \ldots, i_t)$ to a $(k-1)$-long sequence i of distinct integers in $[m]$. Thus, we can rewrite (14) and obtain:

Proposition 5

$$F(k, r, c) = \sum_{i, j} \sum_{t, u \geq 0, t+u < k} \frac{\Pr(r, (i_1, \ldots, i_{k-1})) \Pr(c, (j_1, \ldots, j_{k-1}))}{(R - \sum_{s=1}^{t} r_{i_s})(C - \sum_{s=1}^{u} c_{j_s})}, \tag{15}$$

where the outer sum in (15) is over pairs of ordered sequences of $k-1$ distinct integers from $[m]$ and $[n]$.

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Note that each term in the above sum corresponds to a flag of submatrices of sizes $1 \times 1, \ldots, (k-1) \times (k-1)$. In this form, specializing to the case that all the r's and c's are 1, it is easy to see that Conjecture 2 implies Conjecture 1.

We now group the terms in the outer sum according to the (unordered) sets $I = \{i_1, \ldots, i_{k-1}\}$ and $J = \{j_1, \ldots, j_{k-1}\}$. It then becomes

$$F(k, r, c) = \sum_{I,J} \sum_{i,j} \sum_{t,u \geq 0, t+u < k} \frac{\Pr(r, (i_1, \ldots, i_{k-1})) \Pr(c, (j_1, \ldots, j_{k-1}))}{(R - \sum_{s=1}^{t} r_{i_s})(C - \sum_{s=1}^{u} c_{j_s})} \quad (16)$$

where the outer sum is over sets I and J of size $k-1$ and the inner sum is over permutations $(i_1, \ldots, i_{k-1})$ of I and permutations $(j_1, \ldots, j_{k-1})$ of J. In this form we can see that the term of the outer sum corresponding to the sets I and J is the expected value of the contribution of the submatrix with row indices I and column indices J predicted by Conjecture 3. Since the sum of the expected contributions of all submatrices is the expected minimum k-assignment, we now see that Conjecture 3 implies Conjecture 2.

Finally, for any $T \subseteq I \subseteq [m]$, let $\Pr(r, T, I)$ denote the probability that the first $|T|$ balls drawn from the urn comprise the set T and that the first $|I|$ balls drawn comprise the set I. Then we can rewrite our formula for the expected contribution of a submatrix with rows I and columns J as

$$\sum_{T \subseteq I, U \subseteq J, |T|+|U| < k} \frac{\Pr(r, T, I) \Pr(c, U, J)}{(\sum_{t \notin T} r_t)(\sum_{u \notin U} c_u)} \quad (17)$$

3.2 Flag probabilities

In this section, we prove ap probability result in the special case that the rate matrix has rank 1. This result may be the reason that simple formulas exist for $E_k(A)$ when the matrix A has rank 1.

It is possible for a matrix to have many minimum k-assignments for some k. However, with probability 1, a random matrix X has a unique minimum k-assignment for each k. So, if we let M_k denote a $k \times k$ submatrix of X containing a minimum k-assignment of X, then, with probability 1, M_k is unique. By Lemma 3, the submatrices M_k are nested: $M_1 \subset M_2 \subset \cdots \subset M_k$. We will call this the *flag of submatrices of X*. This flag can also be described by the list $i_1, i_2, \ldots, i_k$ of appended rows and the list $j_1, j_2, \ldots, j_k$ of appended columns. Thus, M_l is the submatrix with rows $i_1, \ldots, i_l$ and columns $j_1, \ldots, j_l$.

It is natural to ask for the probability that a random matrix has a given flag of submatrices. We know of no formula for this probability for general rate matrices. However, for rate matrices of rank 1, we can prove a simple formula for the probability of each flag. Moreover, this formula will involve the probabilities $\Pr(r, (i_1, \ldots, i_k))$ calculated in the previous section.

We first need the following

Lemma 6 *Let X be an exponential random matrix with rank 1 rate matrix $A = (r_i c_j)$. If the minimum $(k-1)$-assignment of X uses the first $k-1$ rows and first $k-1$ columns, then the minimum k-assignment uses an additional row and column. The probability of row i' being the additional row is $\frac{r_{i'}}{\sum_{i=k}^{m} r_i}$, the probability of column j' being the additional column is $\frac{c_{j'}}{\sum_{j=k}^{n} c_j}$, and these events are independent.*

Proof: Let M denote the upper left $(k-1) \times (k-1)$ submatrix of X, and Y the $k \times k$ matrix $\mathrm{Aux}_M(X)$ defined in Section 2.3. Then the upper left submatrix of Y is identical to M, and, by abuse of notation, we will let M denote that submatrix of Y also. If the minimum $(k-1)$-assignment of X lies in M, then by Lemma 4, the minimum $(k-1)$-assignment of Y lies in M, and by Lemma 5, the minimum k-assignments of X and Y correspond entry by entry.

There are two cases to consider.

In the first case, for some $s \leq k-1$ and $t \leq k-1$, the minimum k-assignment of Y uses the entries y_{sk} and y_{kt}. These entries correspond to the minimum entry in row s of X outside of M and the minimum entry in column t of X outside of M. In the second case, the minimum k-assignment of Y uses entry y_{kk}. This entry corresponds to the minimum entry in the submatrix of X complementary to M. In both cases, by Proposition 2, the locations of these minima in X are independent of the random variables making up the entries of Y, and thus independent of the events that the minimum $(k-1)$-assignment of Y lies in M and the minimum k-assignment of Y uses particular entries outside of M.

Thus in the first case, the probability that the minimum entry in the part of row s outside of M comes from column j' is $\frac{r_s c_{j'}}{\sum_{j=k}^{n} r_s c_j} = \frac{c_{j'}}{c_k + \ldots + c_n}$, and the probability that the minimum entry in the part of column t outside of M comes from row i' is $\frac{r_{i'} c_t}{\sum_{i=k}^{m} r_i c_t} = \frac{r_{i'}}{r_k + \ldots r_m}$. Moreover, the locations of these minima within row s and column t are independent events since the parts of row s and column t outside of M are disjoint.

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In the second case, the probability of the minimum entry in the submatrix of X complementary to M coming from row i' and column j' is $\frac{r_{i'} c_{j'}}{\sum r_i c_j}$ where the sum in the denominator is over all locations (i, j) in the submatrix of X complementary to M. Thus, the probability that the minimum entry comes from row i' is $\frac{r_{i'}}{\sum_{i=k}^{m} r_i}$ and the probability that the minimum entry comes from column j' is $\frac{c_{j'}}{\sum_{j=k}^{n} c_j}$. $\square$

From this lemma we can immediately conclude the following theorem, which imparts further meaning to the formal $\Pr(r, I)$ and $\Pr(r, (i_1, \ldots, i_k))$ functions used in Section 3.1:

Theorem 4 *Suppose that $A = (r_i c_j)$ is a rank 1 rate matrix and X an exponential random matrix with rate matrix A. Let $(i_1, \ldots, i_k)$ be a sequence of distinct elements of $[m]$ and $(j_1, \ldots, j_k)$ a sequence of distinct elements from $[n]$. Then the probability that X has the associated flag of submatrices is $\Pr(r, (i_1, \ldots, i_k)) \Pr(c, (j_1, \ldots, j_k))$. Furthermore, if $I \subseteq [m]$ and $J \subseteq [n]$ are sets of size k, then the probability that the minimum k-assignment of X uses the rows indexed by I is $\Pr(r, I)$ and the probability that it uses the columns indexed by J is $\Pr(c, J)$, and these events are independent.*

$\square$

We will see that this formula enters in an essential way into the proof of Theorem 5.

4   Computational evidence for our conjectures

By Proposition 2, it is an easy matter to compute the expected value of the minimum 1-assignment for an arbitrary rate matrix.

Example 4 *The expected value of the minimum 1-assignment when the rate matrix is $A = (a_{ij})$ is*

$$\frac{1}{\sum_{ij} a_{ij}}.$$

For $k \geq 2$ the computation is more complicated. In [AS] and [CS] the authors calculate the expected value of the minimum assignment for a random exponential matrix when the rates are all 1 and k is small.

The method in [AS] applies just as well to the case of arbitrary rate matrices. The essence of their idea is to introduce a slightly more general expectation problem in which they choose all the entries of the random matrix X as before, except that there is a set Z of fixed zeroes in X. Let us denote the expected value of the minimum assignment in this case by $E(A, Z)$.

It is then sometimes possible to establish a recursive calculation of $E(A, Z)$. The base of the recursion occurs when there exist k zeroes in Z, no two in the same row or column. In this case we know that the expected value of the minimum assignment is zero. For the inductive part of the calculation we can sometimes express an expected value $E(A, Z)$ as a constant plus a linear combination of $E(A, Z')$ where Z' is obtained from Z by adjoining one more position to Z.

This arises as follows. Suppose that X is a random exponential matrix except for a set Z of positions in X where the entries are fixed zeroes. Suppose further that we have a set S of positions in X, disjoint from Z, such that any minimum k-assignment of X meets the set S in exactly r positions. (In other words, every nonnegative matrix with zero set Z has the property that its minimum k-assignments all meet S in exactly r positions.) Abusing notation, we also let S denote the matrix which is 1 at the positions in the set S and zero otherwise.

We will derive the following formula:

$$E(A, Z) = \frac{r}{A \cdot S} + \sum_{(i,j) \in S} \frac{a_{ij}}{A \cdot S} E(A, Z \cup \{(i, j)\}). \tag{18}$$

Indeed, the integral for $E(A, Z)$, which involves only the variables x_{ij} for $(i, j) \notin Z$, is given by

$$E(A, Z) = \left(\prod_{(i,j) \notin Z} a_{ij} \right) \int_X \min_k(X) e^{-A \cdot X} dX.$$

We can derive (18) by breaking up this integral into $|S|$ parts, each corresponding to a position in S containing the minimum entry among all positions in S. For the part of the integral where $x_{i_0 j_0}$ is the minimum entry in S, we make a change of variables with Jacobian 1, as follows. We express the x_{ij} in terms of new variables y_{ij} by setting $x_{ij} = y_{i_0 j_0} + y_{ij}$ when $(i, j) \in S - \{(i_0, j_0)\}$ and $x_{ij} = y_{ij}$ otherwise. X can then be written as $Y + y_{i_0 j_0} S$ where Y is a

nonnegative matrix with fixed zeroes at $Z \cup \{(i_0, j_0)\}$. From our hypothesis about S, we have

$$\min_k(X) = \min_k(Y) + ry_{i_0 j_0}.$$

(Otherwise, there would be a non-minimum k-assignment of X, meeting S in fewer than r positions, that becomes a minimum k-assignment of a matrix $X - tS$ for some $t < y_{i_0 j_0}$. But, the matrix $X - tS$ still has zero set Z, so our hypothesis on S would be contradicted.) Thus, this part of the integral becomes

$$\left(\prod_{(i,j) \notin Z} a_{ij} \right) \int_{Y, y_{i_0 j_0}} (\min_k(Y) + ry_{i_0 j_0}) e^{-A \cdot (Y + y_{i_0 j_0} S)} dy_{i_0 j_0} dY.$$

This can be computed as the sum of two integrals in the obvious way. The first is $a_{i_0 j_0} E(A, Z \cup \{(i_0, j_0)\})/(A \cdot S)$ and the second is $r a_{i_0 j_0}/(A \cdot S)^2$. When we sum these expressions over all $(i_0, j_0) \in S$ we obtain (18).

When $k = m = n \leq 4$, it is easy to see that, when we are not in the base case, there always exists a set S of positions in X and disjoint from Z such that every minimum k-assignment of X meets S in the same number of positions.

To illustrate the method we now discuss the case $k = m = n = 4$. First note that if any row or column of X has no fixed zeroes, then we can take that row or column to be the set S. So we can suppose that every row or column has at least one fixed zero.

Now suppose that there is a 3×3 submatrix S of X that has no fixed zeroes. Without loss of generality, we may take this to be the upper left 3×3 submatrix, so the matrix X has the form

$$X = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & . \end{bmatrix} \tag{19}$$

where $*$ means that entry is positive and $.$ means that nothing is known about that entry. Then any minimum 4-assignment must use either two or three entries from S. If it uses three, then it must use x_{44} and some entry x_{ij}, $i, j < 4$. But then we can decrease the value of the assignment by replacing x_{ij} and x_{44} with x_{i4} and x_{4j}, both of which are zero. This

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contradicts the minimality of the 4-assignment we started with. Thus, any minimum 4-assignment must use exactly two entries from S.

If every 3×3 submatrix of X has a fixed zero, and we are not in the base case, then the Hall marriage theorem implies that there is a 2×3 or 3×2 submatrix S that has no fixed zeroes. Suppose the former, which we can take to be the upper left 2×3 submatrix of X. Each of the first three columns has at least one fixed zero. The fixed zeros in those columns must be in more than one row, since every 3×3 submatrix has a fixed zero. Thus, we may assume the matrix X has the form

$$X = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & . & . \\ . & . & 0 & . \end{bmatrix} \tag{20}$$

Any minimum 4-assignment must use one or two entries from S. Suppose a minimum 4-assignment uses two entries from S. It cannot use x_{11}, since then there would be a smaller 4-assignment consisting of $x_{11}, x_{24}, x_{32},$ and x_{43}. Similarly it cannot use x_{12}, x_{21}, or x_{22}. But, it can only use one of x_{13} and x_{23}, so it must use exactly one entry from S.

Thus, we can always find a suitable set S to continue the recursive calculation.

We have used this method to compute the expected minimum k-assignment for various small cases. For the case $k \leq m = n \leq 3$, this is easily carried out by Mathematica and confirms our conjecture.

When $k = m = n = 4$ and $k = m = n = 5$ we were not patient enough to wait for Mathematica to simplify the complete rational expression, even when the rate matrix has rank 1. However, we were able to check that we obtained the correct answer for many random choices or r_i's and c_j's. For this purpose, we used an ordinary C program, but, instead of using exact rational arithmetic, we carried out our calculations modulo a large prime. Even so, the evidence seems to be overwhelming that our conjecture is correct in these cases.

It is possible, although somewhat more complicated, to compute the expected contribution of a $(k-1) \times (k-1)$ submatrix to the expected minimum assignment of a $k \times k$ matrix when $2 \leq k \leq 4$. In the cases $k = 2$ and $k = 3$ we were able to check directly with Mathematica that Conjecture 3 was valid, which proves Conjecture 1 and Conjecture 2 whenever $k \leq 3$.

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When $k = 4$ we obtained computational evidence for the validity of Conjecture 3, checking its validity in a large number of random cases modulo a prime. This provides confirmation of the other conjectures when $k = 4$ and m and n are arbitrary.

5   Additional evidence for the main conjecture

Let $A = (r_i c_j)$ as usual and denote $E_k(A)$ by $E(k, r_1, \ldots, r_m, c_1, \ldots, c_n)$, or simply $E(k, r, c)$. Recall from Section 3.1 the notation $F(k, r, c)$ for the formula in Conjecture 2. In this section we will show that E and F share several properties.

Let us consider how E behaves if we let a collection of the r_i's approach 0. For simplicity we assume that $r_1, r_2, \ldots, r_l$ approach 0. Then the random matrices will have very large entries in the first l rows. When $k \leq m - l$, there are assignments which avoid the first l rows, so in the limit that $r_1, r_2, \ldots, r_l \to 0$, the minimum assignment will avoid those rows and become equal to $E(k, r_{l+1}, \ldots, r_m, c)$.

Now suppose that $k > m - l$. Then a k-assignment must use at least $k - (m - l) = k + l - m$ of the first l rows. But, in our limiting case, these rows will be very large, so the minimum k-assignment will use as few as possible, or exactly $k + l - m$ of them. The contribution of the entries from these rows to the minimum k-assignment will dominate the minimum k-assignment, so in the minimum k-assignment this contribution will be as small as possible. In particular, in the limit as $r_1, r_2, \ldots, r_l \to 0$, this part of the minimum k-assignment will be $E(k + l - m, r_1, \ldots, r_l, c_1, \ldots, c_n)$. By Theorem 4 we know that a set K of $k + l - m$ columns will be used by the part of the assignment in the first l rows with probability $\Pr(c, K)$. When this happens the expected contribution from the remaining rows is $E(m - l, r_{l+1}, \ldots, r_m, c'(K))$ where by $c'(K)$ denotes the c_j's corresponding to columns not in K. Thus, we should have the following

Theorem 5 *When $k \leq m - l$,*

$$\lim_{r_1, \ldots, r_l \to 0} E(k, r, c) = E(k, r_{l+1}, \ldots, r_m, c). \tag{21}$$

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When $k > m - l > 0$,

$$\lim_{r_1,\ldots,r_l \to 0} (E(k, r, c) - E(k + l - m, r_1, \ldots, r_l, c))$$

$$= \sum_K \Pr(c, K) E(m - l, r_{l+1}, \ldots, r_m, c'(K)) \qquad (22)$$

where the sum is over $K \subseteq [n]$ such that $|K| = k + l - m$.

Proof: Let Z be a random exponential $m \times n$ matrix with all entries of mean 1. Then define $X = Z/A$ to be the term by term quotient of the random matrix Z by the fixed rate matrix A, where $a_{ij} = r_i c_j$. Then X is a random exponential matrix with rate matrix A. In particular,

$$E(k, r, c) = E_k(A) = E(\min_k(Z/A)).$$

Let A_u denote the $l \times n$ matrix that is comprised of the first l rows of A, and let A_d denote $(m - l) \times n$ matrix consisting of the last $m - l$ rows of A. Furthermore, let Z_u and Z_d denote random exponential matrices of the same corresponding shapes, again with rate 1.

If $k \leq m - l$, then it is easy to see that

$$\lim_{r_1,\ldots,r_l \to 0} \min_k(Z/A) = \min_k(Z_d/A_d)$$

pointwise almost everywhere (i.e., *almost surely*, as random variables). Since $\min_k(Z/A) \leq \min_k(Z_d/A_d)$, and the expectation of $\min_k(Z_d/A_d)$ is finite, we can apply the dominated convergence theorem to show that the limit of the expectation is equal to the expectation of the limit. Thus,

$$\lim_{r_1,\ldots,r_l \to 0} E(\min_k(Z/A)) = E(\min_k(Z_d/A_d))$$

and therefore

$$\lim_{r_1,\ldots,r_l \to 0} E(k, r, c) = E(k, r_{l+1}, \ldots, r_m, c).$$

Now suppose $k > m - l > 0$. Consider the nonnegative random variable $R = \min_k(Z/A) - \min_{k+l-m}(Z_u/A_u)$. (It is nonnegative because the minimum k-assignment of Z/A must use at least $k + l - m$ elements from Z_u/A_u.) Furthermore,

$$\min_k(Z/A) \leq \min_{k+l-m}(Z_u/A_u) + \max_{m-l}(Z_d/A_d)$$

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where $\max_k X$ denotes the *maximum* k-assignment of a matrix X. Then the second summand is a nonnegative random variable of finite expectation, independent of $r_1, \ldots, r_l$, dominating R. Let χ_K denote the random variable that is 1 or 0 depending upon whether the minimum $(k+l-m)$-assignment of the matrix Z_u/A_u uses precisely the columns from the set K or does not. Then, by Theorem 4, we know that $E(\chi_K) = \Pr(c, K)$, independent of r. Let $C_K(Z_d/A_d)$ denote the submatrix of Z_d/A_d obtained when the columns indexed by K are removed. We can then show

$$\lim_{r_1,\ldots,r_l \to 0} \left(R - \sum_K \chi_K \min_{m-l}(C_K(Z_d/A_d)) \right) = 0$$

with convergence pointwise almost everywhere, where the sum is over $K \subseteq [n]$ such that $|K| = k + l - m$. Furthermore, we can bound the finite sum by a random variable independent of $r_1, \ldots, r_l$. (The random variable $\sum_K \min_{m-l}(C_K(Z_d/A_d))$ will do.) Thus, by the dominated convergence theorem, we can take the limit of the expectations and obtain

$$
\begin{aligned}
\lim_{r_1,\ldots,r_l \to 0} E(R) &= \lim_{r_1,\ldots,r_l \to 0} \sum_K E(\chi_K \min_{m-l}(C_K(Z_d/A_d))) \\
&= \lim_{r_1,\ldots,r_l \to 0} \sum_K E(\chi_K) E(\min_{m-l}(C_K(Z_d/A_d))) \\
&= \sum_K \Pr(c, K) E(m-l, r_{l+1}, \ldots, r_m, c'(K))
\end{aligned}
$$

But, $E(R) = E(k, r, c) - E(k+l-m, r_1, \ldots, r_l, c)$, so we are done.$\square$

We stated Theorem 5 in terms of limits as the first l r's approach 0, in order to simplify the notation. However, since $E(k, r, c)$ is symmetric in the r's and c's, the analogous results hold for any set of l r's approaching 0.

From Example 1, we have that

$$E(1, r_1, \ldots, r_m, c_1, \ldots, c_n) = \frac{1}{(\sum_i r_i)(\sum_j c_j)}. \tag{23}$$

Also when $l = 1$, taking symmetry into account, Theorem 5 reduces to the following.

Suppose that $m > 1$ and $1 \le i \le m$. Then, if $k < m$,

$$\lim_{r_i \to 0} E(k, r, c) = E(k, r_1, \ldots, \hat{r}_i, \ldots, r_m, c) \tag{24}$$

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while if $k = m$,

$$\lim_{r_i \to 0} \left(E(k, r, c) - \frac{1}{r_i \sum_j c_j} \right)$$
$$= \frac{\sum_j c_j E(k-1, r_1, \ldots, \hat{r}_i, \ldots, r_k, c_1, \ldots, \hat{c}_j, \ldots, c_n)}{\sum_j c_j}. \qquad (25)$$

Proposition 6 *There is at most one set of functions*

$$G(k, r_1, \ldots, r_m, c_1, \ldots, c_n),$$

each a linear combination of terms of the form (6), that satisfy the equations (23), (24) and (25) with E replaced by G.

Proof: Let $H(k, m, n) = H(k, r_1, \ldots, r_m, c_1, \ldots, c_n)$ denote the difference between two sets of functions satisfying the conditions of the proposition. We will show by induction on k and m that $H(k, m, n) = 0$.

It is clear that $H(1, m, n) = 0$ for all m and n. Given values of k and m, we may suppose that $H(k', m', n) = 0$ and $H(k, m', n) = 0$ for $k' < k$, $m' < m$, and arbitrary n. Then, by equations (24) and (25) and induction, we have, for any i, $\lim_{r_i \to 0} H(k, m, n) = 0$. Suppose that

$$H(k, m, n) = \sum_I \frac{h_I}{\sum_{i \in I} r_i},$$

where I runs over nonempty subsets of $[m]$ and the h_I's are rational functions of the c's. Suppose that $h_I \neq 0$ for some I. Let I_0 be a minimal I such that $h_I \neq 0$ and let $t \in I_0$. Since $\lim_{r_t \to 0} H(m, n, k)$ exists, we must have $h_{\{t\}} = 0$, so $I_0 \neq \{t\}$. Also

$$\lim_{r_t \to 0} H(k, m, n) = \sum_I \frac{h_I + h_{I-\{t\}}}{\sum_{i \in I-\{t\}} r_i}$$

where the sum is over all I strictly containing $\{t\}$. Since the terms $\frac{1}{\sum_{i \in I} r_i}$ are linearly independent, we must have $h_I + h_{I-\{t\}} = 0$ for all I strictly containing $\{t\}$. In the case $I = I_0$, this contradicts the minimality of I_0. Thus $h_I = 0$ for all I and $H(k, m, n) = 0$. $\square$

Now we show that the rational functions $F(k, r_1, \ldots, r_m, c_1, \ldots, c_n)$ satisfy the same limit conditions that are proved about $E(k, r_1, \ldots, r_m, c_1, \ldots, c_n)$ in Theorem 5.

In particular the functions $F(k, r, c)$ satisfy the conditions of Proposition 6. Thus they are the only possible linear combinations of terms of the form (6) that could equal $E(k, r, c)$.

The fact that the F's satisfy *all* the limit conditions proved about E provides additional evidence for Conjecture 2.

Theorem 6 *When $k \leq m - l$,*

$$\lim_{r_1, \ldots, r_l \to 0} F(k, r, c) = F(k, r_{l+1}, \ldots, r_m, c). \tag{26}$$

When $k > m - l > 0$,

$$\lim_{r_1, \ldots, r_l \to 0} (F(k, r, c) - F(k + l - m, r_1, \ldots, r_l, c))$$
$$= \sum_K \Pr(c, K) F(m - l, r_{l+1}, \ldots, r_m, c'(K)) \tag{27}$$

where the sum is over $K \subseteq [n]$ such that $|K| = k + l - m$.

Proof: We use the alternate form of $F(k, r, c)$ given by (8):

$$F(k, r, c) = \sum_{I,J} \binom{k - 1 - m - n}{k - 1 - |I| - |J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)},$$

where the binomial coefficient enforces the condition $|I| + |J| < k$.

Now, on the left side of (27), before passing to the limit, the first term is

$$\sum_{I,J} \binom{k - 1 - m - n}{k - 1 - |I| - |J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}$$

where the sum is over subsets $I \subseteq [m]$ and $J \subseteq [n]$ and the second term is

$$\sum_{I,J} \binom{k + l - m - 1 - l - n}{k + l - m - 1 - |I| - |J|} \frac{1}{(\sum_{i \notin I, i \leq l} r_i)(\sum_{j \notin J} c_j)}$$

where the sum is over subsets $I \subseteq [l]$ and $J \subseteq [n]$. By substituting $I \cup \{l + 1, \ldots, m\}$ for I in the second term we get

$$\sum_{I,J} \binom{k - 1 - m - n}{k - 1 - |I| - |J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}$$

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where the sum is over subsets $I \subseteq [m]$ and $J \subseteq [n]$ such that $\{l+1, \ldots, m\} \subseteq I$. Thus, before taking the limit, the left side of (27) equals

$$\sum_{I,J} \binom{k-1-m-n}{k-1-|I|-|J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}. \tag{28}$$

where the sum is over all subsets $J \subseteq [n]$ and those subsets $I \subseteq [m]$ that do not contain $\{l+1, \ldots, m\}$.

The last condition on I implies that $\sum_{i \notin I} r_i$ is nonzero if we set $r_1, \ldots, r_l = 0$. Thus, we can obtain the limit on the left of (27) simply by replacing $r_1, \ldots, r_l$ by zero in (28). When we do this, the effect is that we combine terms with I having a fixed intersection with $\{l+1, \ldots, m\}$. Note that this intersection is never all of $\{l+1, \ldots, m\}$.

Suppose then that I is a set strictly contained in $\{l+1, \ldots, m\}$. For each i there are $\binom{l}{i}$ ways of extending I to a $(|I|+i)$-element subset of $[m]$ whose intersection with $\{l+1, \ldots, m\}$ is I. Thus, after taking the limit on the left of (27), we obtain

$$\sum_{I,J} \left(\sum_{i=0}^{l} \binom{l}{i} \binom{k-1-m-n}{k-1-|I|-|J|-i} \right) \frac{1}{(\sum_{i \notin I, i>l} r_i)(\sum_{j \notin J} c_j)}$$

$$= \sum_{I,J} \binom{k+l-1-m-n}{k-1-|I|-|J|} \frac{1}{(\sum_{i \notin I, i>l} r_i)(\sum_{j \notin J} c_j)}$$

where the sum is over proper subsets $I \subsetneq \{l+1, \ldots, m\}$ and all subsets $J \subseteq [n]$. Note that in the case that $k \leq m - l$, this expression is precisely $F(k, r_{l+1}, \ldots, r_m, c)$ so we have proved (26).

We continue with the proof of (27). We obtain a slightly more convenient expression if we replace I by $I \cup [l]$ in the preceding expression. Then the left side becomes

$$\sum_{I,J} \binom{k+l-1-m-n}{k+l-1-|I|-|J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)} \tag{29}$$

where the sum is over sets I strictly contained in $[m]$ and containing $[l]$ and $J \subseteq [n]$.

Now we turn to the right side of (27).

The expression (8) gives $F(m-l, r_{l+1}, \ldots, r_m, c'(K))$ as a sum over certain subsets I of $\{l+1, \ldots, m\}$ and J of $c'(K)$. But this expression is simpler if we replace I by $I \cup [l]$ and J by $J \cup K$. Then the right side can be written

$$\sum_{I \subseteq [m], K \subseteq J \subseteq [n]} \Pr(c, K) \binom{k+l-1-m-n}{k+l-1-|I|-|J|} \frac{1}{(\sum_{i \notin I} r_i)(\sum_{j \notin J} c_j)}$$

where the sum is over proper subsets $I \subsetneq [m]$ containing $[l]$, subsets $K \subseteq J \subseteq [n]$ such that $|K| = k + l - m$.

Now we have shown that both the left and right sides of (27) are linear combinations of the same reciprocal sums $1/(\sum_{i \notin I} r_i)$, so to prove (27) it will suffice to prove that the coefficients of the same reciprocal sums are equal on both sides.

For a given $I \subsetneq [m]$, the coefficient on the left and right depend only on the cardinality of I. We introduce the abbreviations $H = k + l - 1 - |I|$ and $L = k + l - m$. Then $0 < L \leq H < k$. After using these abbreviations and equating coefficients we are reduced to proving

$$\sum_{J \subseteq [n]} \binom{L-n-1}{H-|J|} \frac{1}{\sum_{j \notin J} c_j} = \sum_{K \subseteq J \subseteq [n]} \Pr(c, K) \binom{L-n-1}{H-|J|} \frac{1}{\sum_{j \notin J} c_j} \quad (30)$$

where in the sum on the right the subset K must have cardinality L.

Now use the expression (12) for $\Pr(c, K)$ to rewrite the right side of (30) as

$$\sum_{A \subseteq K \subseteq J \subseteq [n]} \left((-1)^{L-|A|} \frac{\sum_{j \notin K} c_j}{\sum_{j \notin A} c_j} \right) \binom{L-n-1}{H-|J|} \frac{1}{\sum_{j \notin J} c_j}$$

where we still require that $|K| = L$. We sum this first over K. In the term $\sum_{j \notin K} c_j$, the number of K's for which a given c_j occurs depends only on whether j belongs to J. Thus, we can rewrite the right side of (30) as

$$\sum_{A \subseteq J} (-1)^{L-|A|} \left(\frac{\binom{|J|-|A|}{L-|A|} \sum_{j \notin J} c_j + \binom{|J|-|A|-1}{L-|A|} \sum_{j \in J-A} c_j}{\left(\sum_{j \notin A} c_j\right)\left(\sum_{j \notin J} c_j\right)} \right) \binom{L-n-1}{H-|J|}.$$

The numerator of the fraction can be rewritten as

$$\binom{|J| - |A|}{L - |A|} \sum_{j \notin J} c_j + \binom{|J| - |A| - 1}{L - |A|} \sum_{j \notin A} c_j - \binom{|J| - |A| - 1}{L - |A|} \sum_{j \notin J} c_j$$

$$= \binom{|J| - |A| - 1}{L - |A| - 1} \sum_{j \notin J} c_j + \binom{|J| - |A| - 1}{L - |A|} \sum_{j \notin A} c_j$$

so the right side of (30) can be rewritten as a sum of two terms:

$$\sum_A (-1)^{L-|A|} \left(\sum_{J \supseteq A} \binom{|J| - |A| - 1}{L - |A| - 1} \binom{L - n - 1}{H - |J|} \right) \frac{1}{\sum_{j \notin A} c_j} \qquad (31)$$

and

$$\sum_J \left(\sum_{A \subseteq J} (-1)^{L-|A|} \binom{|J| - |A| - 1}{L - |A|} \right) \binom{L - n - 1}{H - |J|} \frac{1}{\sum_{j \notin J} c_j}. \qquad (32)$$

Now, comparing with the left side of (30), it suffices to show that the inner sum in (31) equals $(-1)^{L-|A|} \binom{L-n-1}{H-|A|}$ when $|A| < L$ and 0 otherwise, and the inner sum in (32) equals 1.

It is easy to see that the inner sum of (31) equals 0 when $|A| = L$, since the first binomial coefficient has a negative lower term in that case. When $|A| < L$, because $|J| \geq L$, the inner sum is over J strictly larger than A. Collecting terms according to the cardinality j of J, we obtain

$$\sum_{j=|A|+1}^{H} \binom{n-|A|}{j-|A|} \binom{j-|A|-1}{L-|A|-1} \binom{L-n-1}{H-j}$$

$$= \sum_{i=1}^{H-|A|} \binom{n-|A|}{i} \binom{i-1}{L-|A|-1} \binom{L-n-1}{H-|A|-i}$$

$$= -\binom{n-|A|}{0} \binom{0-1}{L-|A|-1} \binom{L-n-1}{H-|A|-0}$$

$$+ \sum_{i=0}^{H-|A|} \binom{n-|A|}{i} \binom{i-1}{L-|A|-1} \binom{L-n-1}{H-|A|-i}$$

$$= -\binom{-1}{L-|A|-1} \binom{L-n-1}{H-|A|} + \binom{n-L+H-|A|}{H-|A|} \binom{-1}{L-1-H}$$

$$= (-1)^{L-|A|} \binom{L-n-1}{H-|A|}$$

where the third equality holds by substituting into the identity ([R], p.16)

$$\binom{m}{p} \binom{n}{q} = \sum_{i=0}^{p} \binom{n+i}{p+q} \binom{m-n+q}{i} \binom{n-m+p}{p-i}$$

and the fourth equality holds because $L - 1 - H < 0$.

In the inner sum of (32) we can collect terms according to the cardinality a of A. Recalling that A must be contained in J, we obtain

$$\sum_{a} (-1)^{L-a} \binom{|J|}{a} \binom{|J|-a-1}{L-a} = \sum_{a} \binom{|J|}{a} \binom{L-|J|}{L-a} = \binom{L}{L} = 1.$$

This proves Theorem 6. □

Finally we prove that E and F have another property in common.

Theorem 7 *Both $E(k,r,c)$ and $F(k,r,c)$ are monotonically decreasing functions of r and c. In particular, if $r_1 < r_1'$, then*

$$E(k, r_1, r_2, \ldots, r_m, c) > E(k, r_1', r_2, \ldots, r_m, c)$$

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and

$$F(k, r_1, r_2, \ldots, r_m, c) > F(k, r_1', r_2, \ldots, r_m, c).$$

Furthermore, F is differentiable to any degree $\ell \geq 0$ in each r_i and c_j, and $(-1)^\ell \frac{\partial^\ell F}{\partial r_i^\ell} > 0$.

Proof: Without loss of generality, because of symmetry, we can restrict ourselves to considering the behavior of E and F as functions of r_1.

Recall from the proof of Theorem 5 that $E(k, r, c)$ is the expectation of the random variable $\min_k(Z/A)$, where Z is an $m \times n$-matrix-valued random variable with exponentially distributed independent entries of mean 1, where $A = (r_i c_j)$ is the rank 1 rate matrix, and where Z/A denotes the element by element quotient. Suppose $r_1 < r_1'$ and $r_i = r_i'$ for $i = 2, \ldots, m$. Define $A' = (r_i' c_j)$. Then $A \leq A'$ term by term, so $Z/A \geq Z/A'$, and $\min_k(Z/A) \geq \min_k(Z/A')$. Hence,

$$E(k, r, c) = E(\min_k(Z/A)) \geq E(\min_k(Z/A')) = E(k, r', c).$$

Since there is a nonzero probability that the minimum k-assignment of Z/A uses the first row, the inequality is actually strict.

We start with the following formula for F, from (9) and (10):

$$F(k, r, c) = \sum_{|I'|+|J'|<k} \left(\sum_{I \subseteq I'} (-1)^{|I'|-|I|} \frac{1}{\sum_{i \notin I} r_i} \right) \left(\sum_{J \subseteq J'} (-1)^{|J'|-|J|} \frac{1}{\sum_{j \notin J} c_j} \right)$$

For $\ell \geq 1$, $I' \subsetneq [m]$, and $J' \subsetneq [n]$, we define the functions

$$\begin{aligned} f(\ell, r, I') &= \sum_{I \subseteq I'} (-1)^{|I'|-|I|} \frac{1}{(\sum_{i \notin I} r_i)^\ell} \\ g(\ell, c, J') &= \sum_{J \subseteq J'} (-1)^{|J'|-|J|} \frac{1}{(\sum_{j \notin J} c_j)^\ell}, \end{aligned}$$

so that

$$F(k, r, c) = \sum_{|I'|+|J'|<k} f(1, r, I') g(1, c, J'). \tag{33}$$

First we prove that $f(\ell, r, I') > 0$. We define the partial sum $R = \sum_{i \notin I'}^{m} r_i$. Then

$$
\begin{aligned}
0 &< \int_0^\infty t^{\ell-1} e^{-Rt} \left(\prod_{i \in I'} (1 - e^{-r_i t}) \right) dt \\
&= \int_0^\infty t^{\ell-1} \sum_{I \subseteq I'} (-1)^{|I|} \exp(t(-R - \sum_{i \in I} r_i)) dt \\
&= \int_0^\infty t^{\ell-1} \sum_{I \subseteq I'} (-1)^{|I'|-|I|} \exp(t(-\sum_{i \notin I} r_i)) dt \\
&= \sum_{I \subseteq I'} (-1)^{|I'|-|I|} \int_0^\infty t^{\ell-1} \exp(t(-\sum_{i \notin I} r_i)) dt \\
&= (\ell - 1)! \sum_{I \subseteq I'} (-1)^{|I'|-|I|} \frac{1}{(\sum_{i \notin I} r_i)^\ell} \\
&= (\ell - 1)! f(\ell, r, I'),
\end{aligned}
$$

so $f(\ell, r, I') > 0$ as claimed. The proof that $g(\ell, c, J') > 0$ is essentially the same.

To finish the proof of the theorem it suffices to show that for $\ell \geq 1$,

$$
(-\frac{\partial}{\partial r_1})^\ell F(k, r, c) = \sum_{|I'|+|J'|=k-1,\ 1 \notin I'} \ell! f(\ell+1, r, I') g(1, c, J'), \tag{34}
$$

because all terms on the right hand side are positive. First, rewrite (33) as follows

$$
\begin{aligned}
F(k, r, c) &= \sum_{|I'|+|J'|<k-1, 1 \notin I'} (f(1, r, I') + f(1, r, I' \cup \{1\})) g(1, c, J') \\
&+ \sum_{|I'|+|J'|=k-1, 1 \notin I'} f(1, r, I') g(1, c, J').
\end{aligned}
$$

The functions $f(1, r, I') + f(1, r, I' \cup \{1\})$ and $g(1, c, J')$ are independent of r_1, so those partial derivatives vanish. Meanwhile, if $i \notin I'$, then all the denominators in $f(1, r, I')$ involve r_1, so we have

$$
(-\frac{\partial}{\partial r_1})^\ell f(1, r, I') = \ell! f(\ell+1, r, I').
$$

This proves (34) and Theorem 7. □

The functions $E(k, r, c)$ and $F(k, r, c)$ share many properties. Both functions are rational, homogeneous of degree -1, and symmetric in the r_i's and the c_j's. Both rational functions have denominators that factor into linear factors which are either sums of r_i's or sums of c_j's. Both are positive, and both are monotonically decreasing in each r_i and c_j. Finally, E and F share various limit properties with r_i or c_j tending to 0. We could also consider limits as m or n go to ∞. However, it is conceivable that the properties we have already found are sufficient to guarantee that such a function is unique. In any case, one plan to prove $E = F$ would be to extend the list of common properties until equality is forced.

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